Comparison of the Craig-Bampton and Residual Flexibility Methods of Substructure Representation

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A theoretical and numerical comparison is made between the fixed-interface Craig-Bampton method and the free-interface methods of MacNeal and Rubin for component substructure representation. The static and dynamic equivalence of the methods is investigated for a restrained substructure. Vector space theory is used to derive a relation which must be satisfied for dynamic equivalence of the Craig-Bampton and Rubin substructure representations. Numerical comparisons are made using an unrestrained substructure. Synthesized system modes for each of the representations investigated are compared with a reference solution using a mode correlation constant. The effect of the inclusion of rotational degrees of freedom at the component interface is also investigated.

Nomenclature

b	= number of connect degrees of freedom
C	= Craig-Bampton transformation matrix
D	= MacNeal and Rubin transformation matrix
f	= $i+b$, number of degrees of freedom in substructure finite element representation
F_B	= external forces applied at connect degrees of freedom
G_F	= flexibility matrix corresponding to substructure finite element representation
G_N	= flexibility matrix associated with retained free interface normal modes
G_{BB}, G_{IB}	= partitions of G_F
G_ρ	= residual flexibility matrix
$G_{\rho BB}$, $G_{\rho IB}$	= partitions of G_ρ
H	= matrix of constants
i	= number of internal degrees of freedom
K_F	= stiffness matrix corresponding to substructure finite element representation
K_{II}, K_{IB}	= partitions of K_F
K_{NC}, K_{Nr}	= generalized stiffness matrices associated with retained substructure normal modes
n	= number of retained substructure normal modes
q_c, q_r	= generalized degrees of freedom corresponding to retained substructure normal modes
u_F	= displacement vector corresponding to substructure finite element representation
u_{Fc}, u_{Fr}	= vector subspaces of u_F
u_{Ic}, u_{Bc}	= partitions of u_{Fc}
u_{Ir}, u_{Br}	= partitions of u_{Fr}
u_{Sc}, u_{Sr}	= displacement vectors for reduced substructure representations
s	= $n+b$, number of degrees of freedom in reduced substructure representation
ϕ_c	= retained constrained interface substructure normal modes
ϕ_r	= retained free interface substructure normal modes
ϕ_{Ic}	= internal partitions of ϕ_c
ϕ_{Ir}, ϕ_{Br}	= partitions of ϕ_r
ψ_c	= constraint mode matrix
ψ_r	= modified residual attachment modes

Subscripts

B	= vector and matrix partitions associated with connect degrees of freedom
c	= vectors and matrices associated with the Craig-Bampton representation
F	= vectors and matrices associated with substructure finite element representation
I	= vectors and matrix partitions associated with internal degrees of freedom
N	= matrices associated with retained substructure normal modes
q	= matrix partitions associated with retained substructure modal degrees of freedom
r	= vectors and matrices associated with the Rubin and MacNeal representations
S	= vectors and matrices associated with reduced substructure representation
ρ	= matrices associated with truncated free interface substructure normal modes

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I. Introduction

Due to the size and complexity of present day spacecraft and the limited amount of computer resources available, the structural dynamicist must be able to represent an original spacecraft system model by a much reduced set of degrees of freedom. In many instances, the dynamicist responsible for the overall analysis of the system will receive only mathematical models of the system components which have already been reduced to a subset of their original degrees of freedom. Component mode synthesis must then be used to construct the system representation. It is vital that the individual components be dynamically well represented by their reduced mathematical models, or a poor synthesized system solution will result.

Several different methods exist for the representation of a component substructure for the purpose of system mode synthesis. These methods are categorized by the way the connect degrees of freedom are treated during the computation of component normal modes. The two basic categories of methods are 1) fixed-interface and 2) free-interface.

The Craig-Bampton method[1] is the most popular of the fixed-interface techniques. Component normal modes are computed with the connect degrees of freedom fixed, and the retained normal modes are augmented with a set of "constraint" modes. For many applications, this technique yields very accurate results for a relatively small number of retained component modes. However, the use of fixed-interface methods in conjunction with modal testing presents difficulties not found in the use of free-interface methods[2] of mode synthesis. Free-interface techniques use component normal modes which are computed with the connect degrees of freedom unconstrained. In order to obtain sufficient accuracy in the system solution, the normal modes must be supplemented with "attachment" modes.

The residual flexibility method is a free-interface technique which was first developed by MacNeal[3] to include the static effects of higher-order normal modes not retained in the component representation. MacNeal's method has a distinct advantage over other free-interface techniques in that the retained component normal modes and the residual attachment modes are always linearly independent. The inclusion of the residual flexibility of higher-order modes is especially important when rigidly connecting flexible components.

Rubin[4] extended MacNeal's method to include the inertial effects of higher order modes by using a second order Maclaurin-series expansion in the Laplace variable. Martinez et al.[5] identified Rubin's method as a consistent Ritz transformation using "modified" free-interface elastic modes and "modified" residual attachment modes as trial vectors. This paper compares the Craig-Bampton fixed-interface method with the free-interface MacNeal and Rubin residuals methods for component substructure representation. The theoretical formulations of the substructure equations for each method are compared for a restrained substructure. A numerical comparison of the methods is made for an unrestrained substructure. The effect of including rotational degrees of freedom at the connection points is also investigated for each method. Similar theoretical comparisons of substructure synthesis methods have been attempted by Craig and Chang[6] and Craig.[7]

II. Theoretical Comparison

The substructure is represented by a finite element model which contains a large number of physical degrees of freedom u_F. The equation of motion for the undamped substructure can be written as

$$M_F\ddot{u}_F + K_F u_F = F_F \tag{1}$$

where M_F and K_F are the physical mass and stiffness matrices, respectively. Usually, the vector u_F will contain more degrees of freedom than can be handled economically. Therefore, a transformation is used to reduce the number of degrees of freedom representing the substructure to a more manageable size. If a transformation of the form $u_F = Tu_S$ is used, where u_S is the truncated generalized displacement vector of the substructure, the equation of motion for the reduced representation is given by

$$M_S\ddot{u}_S + K_S u_S = F_S \tag{2}$$

where

$$M_S = T^T M_F T; \qquad K_S = T^T K_F T; \qquad F_S = T^T F_F \tag{3}$$

Normal modes of the substructure can be used in a Rayleigh-Ritz transformation to accomplish the reduction. However, in the case of modal synthesis, if a very flexible component substructure is to be rigidly attached to another substructure, the normal modes must be augmented with static displacement shapes to account for flexibility lost at the connection points due to modal truncation.

The Craig-Bampton method uses the Ritz transformation

$$u_{Fc} = Cu_{Sc} \tag{4}$$

where

$$u_{Fc} = [u_{Ic}{}^T{:}u_{Bc}{}^T]^T, \; u_{Sc} = [q_c{}^T{:}u_{Bc}{}^T]^T, \; C = [\phi_c{:}\psi_c]$$

$$\phi_c = [\phi_{Ic}{}^T{:}0^T]^T, \; \psi_c = [(-K_{II}{}^{-1}K_{IB})^T{:}I]^T$$

A column of the constraint mode matrix ψ_c represents the static deflection shape derived by applying a unit displacement to one of the connect degrees of freedom while restraining the others. Constraint modes are derived by solving the equation

$$K_F[\psi_{Ic}{}^T{:}I]^T = [0^T{:}F_B{}^T]^T \tag{5}$$

yielding $\psi_{IC} = -K_{II}{}^{-1}K_{IB}$. The Craig-Bampton representation uses the combined sets ϕ_c and ψ_c as trial vectors in a consistent Ritz transformation where both of the substructure matrices M_F and K_F are transformed according to Eq. (3). Due to the linear independence of the columns of C, the transformation represents a linear one-to-one mapping from the s dimensional vector space R^s into the f dimensional vector space R^f. The range of C is given by the vector space u_{Fc}, which is a subspace of R^f of dimension s spanned by the columns of C.

MacNeal's method of residual flexibility uses the Ritz transformation

$$u_{Fr} = Du_{Sr} \tag{6}$$

where $D = [\phi_r{}^m{:}\psi_r]$. Martinez et al.[5] identified $\phi_r{}^m$ as a set of "modified" normal modes and ψ_r as a set of "modified" residual attachment modes given by

$$\begin{aligned} \phi_r{}^m &= [(\phi_{Ir} - G_{\rho IB}G_{\rho BB}{}^{-1}\phi_{Br})^T{:}0^T]^T \\ \psi_r &= [(G_{\rho IB}G_{\rho BB}{}^{-1})^T{:}I]^T \end{aligned} \tag{7}$$

The residual flexibility matrix G_ρ defined by the equation

$$G_\rho = G_F - G_N \tag{8}$$

represents the portion of the flexibility matrix associated with the truncated substructure normal modes where

$$G_N = \phi_r K_{Nr}{}^{-1}\phi_r{}^T \text{ and } \phi_r = [\phi_{Ir}{}^T{:}\phi_{Br}{}^T]^T$$

Details of the derivation of the residual flexibility matrix can be found in Refs. 4 and 5.

For equal numbers of substructure physical degrees of freedom, retained normal modes, and connect degrees of freedom, Eq. (6) also represents a linear one-to-one mapping from R^s into R^f. However, MacNeal's method is not a consistent transformation. The transformation is applied only to the substructure stiffness matrix K_F. The mass matrix is not transformed; thus the inertial effects of the static mode shapes ψ_r are neglected. The reduced hybrid substructure mass matrix is formed by augmenting the generalized mass matrix M_N with null partitions. Due to the inconsistent use of the Ritz transformation, MacNeal's method will never be equivalent to the Craig-Bampton method for a dynamics problem and, in general, will not be as complete a representation of the substructure as the Craig-Bampton representation.

The Rubin representation can be derived using the MacNeal transformation of Eq. (6) in a consistent manner. Both the mass and stiffness matrices are transformed according to Eq. (3) yielding a hybrid substructure representation which is identical to the equations derived by Rubin[4] using a different approach. Therefore, the consistent use of the Ritz transformation [Eq. (6)] is equivalent to the second order Maclaurin-series expansion employed by Rubin.

The Craig-Bampton and Rubin methods both use consistent Ritz transformations representing linear one-to-one mappings of R^s into R^f to reduce the order of the eigenvalue problem. Therefore, there may be certain conditions under which the two methods are equivalent. The ranges of the transformations C and D, u_{Fc} and u_{Fr}, respectively, are subspaces of R^f with dimension s. Any vector in either of the subspaces can be written as a linear combination of the columns of its associated transformation matrix. If the two methods are equivalent, the subspaces u_{Fc} and u_{Fr} must be identical. This implies that each of the basis vectors of u_{Fc} (columns of C) must be expressible as a linear combination of the basis vectors of u_{Fr} (columns of D) and vice versa. The two methods will then be equivalent if and only if

$$C = DH \tag{9}$$

where H is a nonsingular $s \times s$ matrix of constants. Transformation D would then represent a linear one-to-one mapping of R^s onto u_{Fc}.

III. Statics Problem

Before proceeding with an investigation of Eq. (9), the statics problem will be examined for both the Craig-Bampton and Rubin substructure representations. The substructure is assumed to be restrained such that rigid body motion is eliminated. External forces are assumed to be applied only at the connect degrees of freedom.

Using a Craig-Bampton representation, the statics problem is of the form

$$\begin{bmatrix} K_{Nc} & 0 \\ 0 & K_{BB} - K_{BI}K_{II}^{-1}K_{IB} \end{bmatrix} \begin{bmatrix} q_c \\ u_{Bc} \end{bmatrix} = \begin{bmatrix} 0 \\ F_B \end{bmatrix} \tag{10}$$

The solution to Eq. (10) is given by

$$q_c = 0 \text{ and } u_{Bc} = [K_{BB} - K_{BI}K_{II}^{-1}K_{IB}]^{-1}F_B \tag{11}$$

Equation (4) is used to recover the internal partition of the displacement vector u_{Ic}, yielding

$$u_{Ic} = K_{II}^{-1}K_{IB}u_{Bc} \tag{12}$$

The Craig-Bampton method is statically complete[8]; therefore, Eqs. (11) and (12) give the exact solution to the statics problem.

Rubin's formulation produces a statics problem of the form

$$\begin{bmatrix} K_{Nr} + \phi_{Br}^T G_{\rho BB}^{-1}\phi_{Br} & -\phi_{Br}^T G_{\rho BB}^{-1} \\ -G_{\rho BB}^{-1}\phi_{Br} & G_{\rho BB}^{-1} \end{bmatrix} \begin{bmatrix} q_r \\ u_{Br} \end{bmatrix} = \begin{bmatrix} 0 \\ F_B \end{bmatrix} \tag{13}$$

The first equation in Eq. (13) can be solved for q_r, yielding

$$q_r = K_{Nr}^{-1}\phi_{Br}^T F_B \tag{14}$$

Equation (14) gives the generalized displacements of the retained modes due to a set of external forces applied at the connect degrees of freedom. This result can be substituted into the second equation of the Eq. (13) to yield

$$u_{Br} = [\phi_{Br}K_{Nr}^{-1}\phi_{Br}^T + G_{\rho BB}]F_B \tag{15}$$

Table 1 Reference normal mode solution for total system finite element model

Percent strain energy summary		Entity	
Eigenvector	Frequency, Hz	Cradle	Satellite
1	1.0755	82.52	17.48
2	1.2274	17.33	82.67
3	1.3701	40.44	59.56
4	1.4385	34.31	65.69
5	2.1191	75.69	24.31
6	3.0085	72.02	27.98
7	3.0507	79.83	20.17
8	4.8460	67.04	32.96
9	6.1980	78.88	21.12

The term $\phi_{Br}K_{Nr}^{-1}\phi_{Br}^T$ is equivalent to the flexibility matrix G_N partitioned to the connect degrees of freedom. Therefore, Eq. (15) can be written as

$$u_{Br} = G_{BB}F_B \tag{16}$$

where G_{BB} is the connect degrees of freedom partition of the flexibility matrix G_F.

The internal partition of u_{Fr} is recovered using Eq. (6), yielding

$$u_{Ir} = [\phi_{Ir} - G_{\rho IB}G_{\rho BB}^{-1}\phi_{Br}]q_r + G_{\rho IB}G_{\rho BB}^{-1}u_{Br} \tag{17}$$

Substituting Eqs. (14) and (16) into Eq. (17) gives

$$u_{Ir} = [G_{NIB} + G_{\rho IB}G_{\rho BB}^{-1}(G_{BB} - G_{NBB})]F_B \tag{18}$$

which yields

$$u_{Ir} = G_{IB}G_{BB}^{-1}u_{Br} \tag{19}$$

By definition $K_F G_F = I$; therefore,

$$G_{BB} = [K_{BB} - K_{BI}K_{II}^{-1}K_{IB}]^{-1} \tag{20}$$

and

$$G_{IB} = -K_{II}^{-1}K_{IB}G_{BB} \tag{21}$$

Equations (16) and (18–21) can be combined to yield

$$u_{Br} = u_{Bc}, \quad u_{Ir} = u_{Ic}$$

Therefore, the Craig-Bampton and Rubin methods of substructure representation are statically equivalent. However, when external forces are applied only at the connect degrees of freedom, the statics problem is a degenerate case in that the vector space u_F is spanned by only b basis vectors instead of the s vectors required to span u_F for a general dynamics problem. In the Craig-Bampton formulation, the b vectors are the constraint modes ψ_c. In Rubin's representation, the columns $[(G_{IB}G_{BB}^{-1})^T{:}I]^T$ form a basis for u_F.

IV. Equivalence Requirements

For the general dynamics problem, the two methods will be equivalent only if Eq. (9) is satisfied. If the matrices C, D, and H are divided into their respective I, B, and q partitions, Eq. (9) yields the following four relations:

$$C_{Iq} = D_{Iq}H_{qq} + D_{IB}H_{Bq} \tag{22}$$

$$C_{IB} = D_{Iq}H_{qB} + D_{IB}H_{BB} \tag{23}$$

$$C_{Bq} = D_{Bq}H_{qq} + D_{BB}H_{Bq} \tag{24}$$

$$C_{BB} = D_{Bq}H_{qB} + D_{BB}H_{BB} \tag{25}$$

Equations (24) and (25) can be solved to yield $H_{Bq}=0$ and $H_{BB}=I$. Substituting these results into Eqs. (22) and (23) gives

$$\phi_{Ic}=[\phi_{Ir}-G_{\rho IB}G_{\rho BB}^{-1}\phi_{Br}]H_{qq} \tag{26}$$

and

$$-K_{II}^{-1}K_{IB}=[\phi_{Ir}-G_{\rho IB}G_{\rho BB}^{-1}\phi_{Br}]H_{qB}+G_{\rho IB}G_{\rho BB}^{-1} \tag{27}$$

Equations (14), (16), and (17) from the Rubin statics problem investigation can be combined to give

$$u_{Ir}=[(\phi_{Ir}-G_{\rho IB}G_{\rho BB}^{-1}\phi_{Br})K_{Nr}^{-1}\phi_{Br}^{T}+G_{\rho IB}G_{\rho BB}^{-1}G_{BB}]F_{B} \tag{28}$$

Equations (11) and (12) from the investigation of the Craig-Bampton statics problem yield

$$u_{Ic}=-K_{II}^{-1}K_{IB}[K_{BB}-K_{BI}K_{II}^{-1}K_{IB}]^{-1}F_{B} \tag{29}$$

In comparing the two static solutions of Sec. III, it was found that $u_{Ir}=u_{Ic}$. Therefore, Eqs. (28) and (29) can be combined to form the relation

$$[\phi_{Ir}-G_{\rho IB}G_{\rho BB}^{-1}\phi_{Br}]K_{Nr}^{-1}\phi_{Br}+G_{\rho IB}G_{\rho BB}^{-1}G_{BB}=-K_{II}^{-1}K_{IB}[K_{BB}-K_{BI}K_{II}^{-1}K_{IB}]^{-1} \tag{30}$$

Postmultiplication of Eq. (30) by G_{BB}^{-1} and use of Eq. (20) yield

$$[\phi_{Ir}-G_{\rho IB}G_{\rho BB}^{-1}\phi_{Br}]K_{Nr}^{-1}\phi_{Br}^{T}[K_{BB}-K_{BI}K_{II}^{-1}K_{IB}]+G_{\rho IB}G_{\rho BB}^{-1}=-K_{II}^{-1}K_{IB} \tag{31}$$

Comparing Eqs. (27) and (31) reveals that the two methods satisfy Eq. (27) with

$$H_{qB}=K_{Nr}^{-1}\phi_{Br}^{T}[K_{BB}-K_{BI}K_{II}^{-1}K_{IB}]$$

Therefore, the methods will be equivalent if and only if the internal degrees-of-freedom partitions of the constrained interface normal modes ϕ_{Ic} can be expressed as linear combinations of the modified free-interface normal modes ϕ_{Ir}^{m}. This condition is expressed by Eq. (26), where H_{qq} is a nonsingular matrix of constants and i is assumed to be greater than n.

The vectors $G_{\rho IB}G_{\rho BB}^{-1}$ in Eq. (26) represent the internal partitions of the modified residual attachment modes. If K_ρ represents the residual stiffness matrix, Eq. (21) implies that for residuals

$$G_{\rho IB}=-K_{\rho II}^{-1}K_{\rho IB}G_{\rho BB} \tag{32}$$

which yields

$$G_{\rho IB}G_{\rho BB}^{-1}=-K_{\rho II}^{-1}K_{\rho IB} \tag{33}$$

Therefore, the modified residual attachment modes ψ_r are actually residual constraint modes which represent the static deflections of the substructure due to the residual portion of the stiffness matrix K_ρ resulting from a unit displacement at each of the connect degrees of freedom. Vectors $G_{\rho IB}G_{\rho BB}^{-1}\phi_{Br}$ then represent static deflections of the internal degrees of freedom due to unit displacements of the connect partitions of the retained free-interface normal modes.

The internal partitions of the modified normal modes ϕ_{Ir}^{m} are thus represented by a set of free-interface normal mode internal partitions minus a set of constraint mode internal partitions. Modified normal modes ϕ_{Ir}^{m} therefore tend to approximate the internal partitions of a set of modes constrained at the connect degrees of freedom which, depending upon the application, may be expressible as linear combinations of the internal partitions of the retained fixed-interface normal modes ϕ_{Ic} used in the Craig-Bampton representation. A unique solution to Eq. (26) would then exist and the methods would be equivalent.

V. Numerical Comparison

The question of the dynamic equivalence of substructure representation methods is of extreme importance to the dynamics analyst, who is constantly concerned with both accuracy and cost efficiency. If the fixed- and free-interface methods are, in general, dynamically equivalent, the analyst can use a single method for a wide variety of problems with confidence. If the methods are not dynamically equivalent, care must be taken to choose the method which maximizes accuracy and minimizes cost for each particular problem being solved.

The equivalency of the Rubin and Craig-Bampton methods is examined using the numerical example illustrated in Fig. 1. The system is divided into two components: an unrestrained flexible cylindrical satellite and a restrained cradle which is very stiff in comparison to the satellite interface. Due to the dissimilarity in the flexibility at the interface between the satellite and the cradle, a classical free-interface representation of the satellite will yield poor results in the

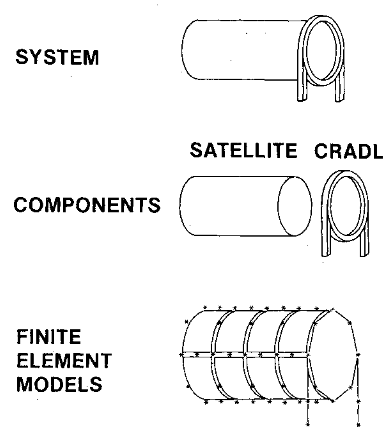

Fig. 1 Example system used for numerical comparison of substructure representation methods.

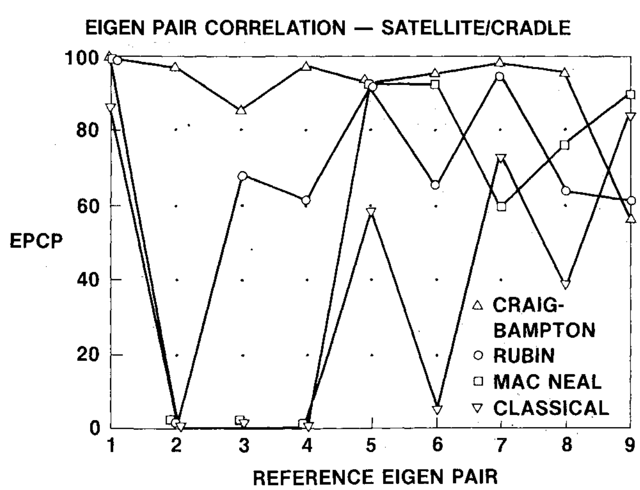

Fig. 2 Comparison of substructure representations for translational connections.

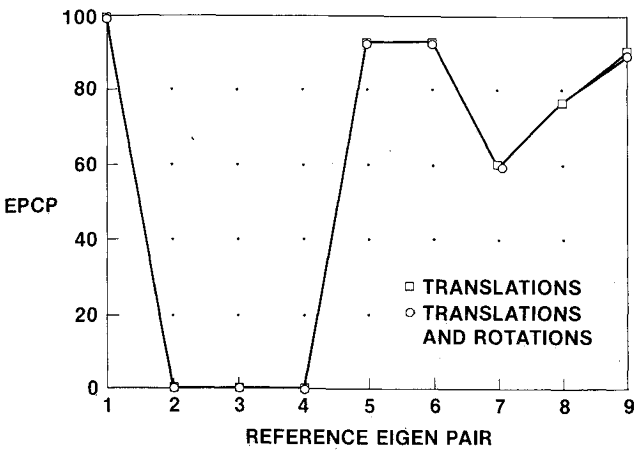

Fig. 3 Effects of rotations at connections for MacNeal's method.

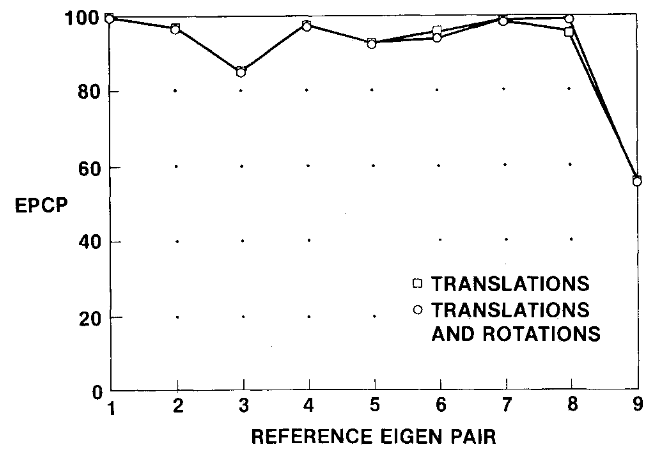

Fig. 5 Effects of rotations at connections for Craig-Bampton method.

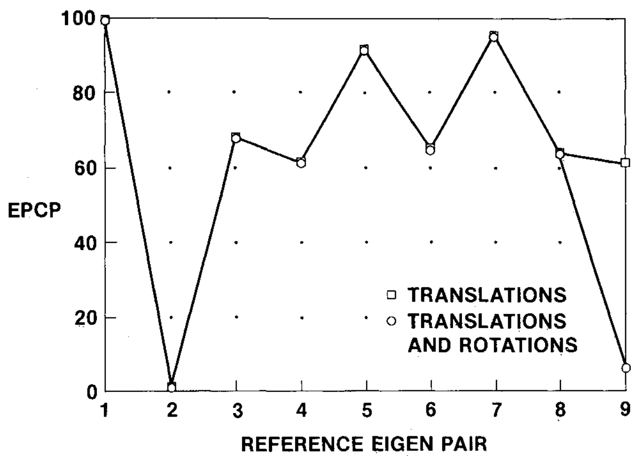

Fig. 4 Effects of rotations at connections for Rubin's method.

synthesis of the system modes. A classical free-interface representation is here defined to be a substructure representation where only free-interface component normal modes are used as Ritz trial vectors.

This example was chosen because it represents a severe test of the general equivalency of the methods for equal numbers of component mode shapes. The fixed-interface component modes used by the Craig-Bampton method more closely describe the behavior of the satellite in the system modes than do the free-interface modes used by the classical, MacNeal, and Rubin methods. While severe, the example is very representative of aerospace structures such as a flexible satellite rigidly fixed to a stiff cradle in the Shuttle.

Finite element models (Fig. 1) were constructed for each of the components and the total system. The satellite contained sixteen thin shell parabolic elements, and the cradle contained twelve linear beam elements. Initially, a normal-modes analysis was performed using the total finite element model of the system to obtain a reference solution for comparison with the substructure representation methods being investigated. Frequencies and strain energy distributions for the first nine system modes are listed in Table 1.

Satellite component substructure matrices were generated using each of the methods of MacNeal, Rubin, Craig-Bampton, and classical free-interface. In each case, the first ten normal modes of the satellite (including six rigid body modes for free-interface methods) were included in the set of trial vectors. The cradle component was always represented using the classical free-interface method including twenty component normal modes. Because the cradle is so stiff compared to the satellite, the classical method results in an accurate representation of the cradle dynamics.

System modes were first synthesized for each of the satellite representations connecting only the three translational degrees of freedom at the eight interface points between the components. The synthesized system modes and frequencies were compared to the reference solution using an eigen pair correlation product (EPCP) defined by the relation

$$\text{EPCP} = \frac{|\phi_{\text{APP}}{}^T\phi_{\text{REF}}|^2 e^{-|\omega_{\text{APP}}-\omega_{\text{REF}}|} \cdot 100}{[\phi_{\text{APP}}{}^T\phi_{\text{APP}}][\phi_{\text{REF}}{}^T\phi_{\text{REF}}]} \tag{34}$$

where ϕ_{APP}, ω_{APP}, ϕ_{REF}, and ω_{REF} are the approximate synthesized system modes and frequencies and reference system modes and frequencies, respectively. This correlation product is just the Modal Assurance Criterion[9] with a penalty for frequency error. If an approximate synthesized mode shape/frequency pair agrees perfectly with one of the reference pairs, EPCP will have a value of 100 for that particular reference pair. The synthesized solution therefore correctly predicts the existence of that particular normal mode.

Results from the four system mode synthesis analyses for translation connections are presented in Fig. 2. The classical free-interface solution, as predicted, gives very poor results for the first nine system modes. Reference modes two, three, and four are not even predicted. MacNeal's method accounts for the flexibility of the normal modes truncated (residual flexibility) from the satellite representation in the classical free-interface analysis. The synthesized solution predicts the first reference mode very accurately. Reference modes five, six, and nine are predicted with less accuracy. Modes two, three, and four remain unpredicted using MacNeal's method.

The consistent transformation of Rubin includes inertial effects of the satellite neglected by MacNeal's representation. Modes three and four are now predicted by the synthesized solution; mode two is still undetected. For this particular example, Rubin's method gives a better overall solution for the first nine system modes than MacNeal's method. However, the synthesized results still differ significantly from the reference solution. The Craig-Bampton satellite representation, which uses constrained interface normal modes, yields good synthesized results for the first eight reference modes as illustrated in Fig. 2.

The inability of the free-interface methods to predict the second, third, and fourth system mode shapes is due to a combination of severe truncation of the free-interface modal set used in the methods for the satellite representation and the severity of the frequency error penalty included in the correlation analysis. Each of the modes which were not predicted by the classical and MacNeal methods contained a high percentage of overall strain energy in the satellite as indicated in Table 1. Rubin's method picked up modes three and four but failed to predict the second mode because it is totally dominated by the deformation of the satellite. These results indicate that 10 free-interface mode shapes do not adequately represent the flexibility of the satellite in the system modes even when residual mass and stiffness effects are included. If the number of free-interface satellite mode shapes is increased to a total of 20 so that the highest component modal frequency is a factor of 1.5 greater than the highest desired system mode, Rubin's method yields accurate results for this example.

In Sec. III, it was shown that the residual flexibility method of Rubin is statically complete. Therefore, the stiffness of the truncated substructure normal modes is represented exactly by the modified residual attachment modes included in the substructure representation. However, the inertial properties of the truncated modes are not totally accounted for. Rubin's method includes only the inertial properties associated with the residual attachment modes which are static deflection shapes of the substructure. Depending upon the application, these static shapes may or may not give a good approximation to the inertial properties of the truncated normal modes. Therefore, if the retained normal modes (free-interface, in the case of Rubin) give a poor approximation of the shape of the substructure in a connected system, the inertial properties of the substructure will be poorly approximated and may or may not be improved by including the inertial properties of the residual attachment modes. The Craig-Bampton method suffers from the same type of problem using constraint modes. The difference in the accuracy of the Rubin and Craig-Bampton methods is thus principally determined by the inertial completeness of their respective retained normal mode sets.

System normal modes were also synthesized using each of the substructure representation methods with both translations and rotations connected at the interface between the two substructures. The effects of the rotational degrees of freedom are presented in Figs. 3-5. In each case, connecting the rotational degrees of freedom had essentially no impact upon the synthesized solution for the first eight system modes. This result supports the theory implied by Martinez et al.[5] that rotational information at the substructure interface is not required when the substructure possesses a large number of connect degrees of freedom.

VI. Conclusion

The Craig-Bampton and residual flexibility methods of component substructure representation were compared both theoretically and numerically. A theoretical comparison for a restrained substructure showed that the MacNeal, Rubin, and Craig-Bampton methods are statically complete. The methods, therefore, represent the stiffness of the structures exactly by including the stiffness associated with truncated normal modes in the form of static displacement vectors.

The Rubin and Craig-Bampton methods were identified as consistent Ritz transformations which include the inertial properties of the static displacement vectors used in each representation in conjunction with normal modes. A relation was derived using vector space theory, which showed that the Craig-Bampton and Rubin methods are equivalent if and only if the modified normal mode internal partitions ϕ_{Ir}^{m} used in Rubin's method are spanned by the internal partitions of the fixed-interface normal modes ϕ_{Ic} used in the Craig-Bampton method. MacNeal's method is an inconsistent Ritz transformation and therefore cannot be dynamically equivalent to the Craig-Bampton method.

The numerical example investigated in this paper indicates that the Rubin and Craig-Bampton methods generally are not equivalent for equal numbers of component modes, indicating that the Ritz vectors incorporated in each method generally do not span the same vector space. The authors wish to emphasize that in no way do the results of the numerical comparison indicate that the Craig-Bampton method is in general a superior method for substructure representation. For each particular problem, the analyst must carefully consider how a substructure will deform in the complete system mode shapes before selecting a substructure representation method.

References

[1]Craig, R.R., Jr. and Bampton, M.C.C., "Coupling of Substructures for Dynamic Analysis," *AIAA Journal*, Vol. 6, 1968, pp. 1313–1319.

[2]Baker, M., "Component Mode Synthesis Methods for Test-Based, Rigidly Connected, Flexible Components," AIAA/ASME/SAE Structures, Structural Dynamics and Materials Conference, Palm Springs, CA, May 14–16, 1984.

[3]MacNeal, R.H., "A Hybrid Method of Component Mode Synthesis," *Computers and Structures*, 1, 1971, pp. 581–601.

[4]Rubin, S., "An Improved Component-Mode Representation," *Proceedings of the AIAA/ASME/SAE 15th Structures, Structural Dynamics and Materials Conference*, Las Vegas, NV, April 17–19, 1974.

[5]Martinez, D.R., Carne, T.G., and Miller, A.K., "Combined Experiential/Analytical Modeling Using Component Mode Synthesis," AIAA Paper 84-0941, May 1984.

[6]Craig, R.R., Jr., and Chang, C.J., "On the Use of Attachment Modes in Substructure Coupling for Dynamic Analysis," *Proceedings* of the AIAA/ASME 18th Structures, Structural Dynamics and Materials Conference, Vol. B, 1977, pp. 89–99.

[7]Craig, R.R., Jr., "A Review of Time-Domain and Frequency-Domain Component Synthesis Methods," *Combined Experimental/Analytical Modeling of Dynamic Structural Systems*, AMD-Vol. 67, 1985, pp. 1–30.

[8]Hintz, R.M., "Analytical Methods in Component Modal Synthesis," *AIAA Journal*, Vol. 13, Aug. 1975, pp. 1007–1016.

[9]Ewins, D.J., *Model Testing: Theory and Practice*, Research Studies Press Ltd., Letchworth, England, 1984, p. 225.